

Finding the growth rate of a regular language in polynomial time

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Abstract

We give an $O(n^3 + n^2t)$ time algorithm to determine whether an NFA with n states and t transitions accepts a language of polynomial or exponential growth. We also show that given a DFA accepting a language of polynomial growth, we can determine the order of polynomial growth in quadratic time.

1 Introduction

Let $L \subseteq \Sigma^*$ be a language. If there exists a polynomial $p(x)$ such that $|L \cap \Sigma^m| \leq p(m)$ for all $m \geq 0$, then L has *polynomial growth*. Languages of polynomial growth are also called *sparse* or *poly-slender*.

If there exists a real number $r > 1$ such that $|L \cap \Sigma^m| \geq r^m$ for infinitely many $m \geq 0$, then L has *exponential growth*. Languages of exponential growth are also called *dense*.

If there exist words $w_1, w_2, \dots, w_k \in \Sigma^*$ such that $L \subseteq w_1^* w_2^* \cdots w_k^*$, then L is called a *bounded language*.

Ginsburg and Spanier [6] (see Ginsburg [5, Chapter 5]) proved many deep results concerning the structure of bounded context-free languages. One significant result [5, Theorem 5.5.2] is that determining if a context-free grammar generates a bounded language is decidable. However, although it is a relatively straightforward consequence of their work, they did not make the following connection between the bounded context-free languages and those of polynomial growth.

Theorem 1. *A context-free language is bounded if and only if it has polynomial growth.*

Curiously, this result has been independently discovered at least six times: namely, by Trofimov [17], Latteux and Thierrin [11], Ibarra and Ravikumar [14], Raz [15], Incitti [9], and Bridson and Gilman [2]. A consequence of all of these proofs is that a context-free language has either polynomial or exponential growth; no intermediate growth is possible.

The particular case of the bounded regular languages was also studied by Ginsburg and Spanier [7], and subsequently by Szilard, Yu, Zhang, and Shallit [16] (see also [8]). It follows from the more general decidability result of Ginsburg and Spanier that there is an algorithm to determine whether a regular language has polynomial or exponential growth (see also [16, Theorem 5]). Ibarra and Ravikumar [14] observed that the algorithm of Ginsburg and Spanier runs in polynomial time for NFA's, but they gave no detailed analysis of the runtime. Here we specialize the algorithm of Ginsburg and Spanier to the case of regular languages, and we give particular attention to the runtime of this algorithm. We also show how, given a DFA accepting a language of polynomial growth as input, one may determine the precise order of polynomial growth in polynomial time.

2 Polynomial vs. exponential growth

In this section we give an $O(n^3 + n^2t)$ time algorithm to determine whether an NFA with n states and t transitions accepts a language of polynomial or exponential growth.

Theorem 2. *Given a NFA M , it is possible to test whether $L(M)$ is of polynomial or exponential growth in $O(n^3 + n^2t)$ time, where n and t are the number of states and transitions of M respectively.*

Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA. We assume that every state of M is both accessible and co-accessible, i.e., every state of M can be reached from q_0 and can reach a final state. For each state $q \in Q$, we define a new NFA $M_q = (Q, \Sigma, \delta, q, \{q\})$ and write $L_q = L(M_q)$.

Following Ginsburg and Spanier, we say that a language $L \subseteq \Sigma^*$ is *commutative* if there exists $u \in \Sigma^*$ such that $L \subseteq u^*$.

The following two lemmas have been obtained in more generality in all of the previously mentioned proofs of Theorem 1 (compare also Lemmas 5.5.5 and 5.5.6 of Ginsburg [5], or in the case of regular languages specified by DFA's, Lemmas 2 and 3 of Szilard et al. [16]).

Lemma 3. *If $L(M)$ has polynomial growth, then for every $q \in Q$, L_q is commutative.*

Proof. A classical result of Lyndon and Schützenberger [12] implies that if a set of words X does not satisfy $X \subseteq u^*$ for any word u , then there exist $x, y \in X$ such that $xy \neq yx$. Suppose then that $L(M)$ has polynomial growth, but for some L_q there exists $x, y \in L_q$, $xy \neq yx$. Let v be any word such that $q \in \delta(q_0, v)$, and let v' be any word such that $\delta(q, v') \cap F \neq \emptyset$. Then for every $m \geq 0$, the set $v(xy + yx)^m v'$ consists of 2^m distinct words of length $|vv'| + m|xy|$ in $L(M)$. It follows that $L(M)$ has exponential growth, contrary to our assumption. \square

Lemma 4. *If for every $q \in Q$, L_q is commutative, then $L(M)$ has polynomial growth.*

Proof. We prove by induction on the number n of states of M that the hypothesis of the lemma implies that $L(M)$ is bounded. It is well-known that any bounded language has polynomial growth (see, for example, [15, Proposition 1]). Clearly the result holds for $n = 1$. We suppose then that $n > 1$.

Let $Q' = Q \setminus \{q_0\}$, $F' = F \setminus \{q_0\}$, and $\delta'(q, a) = \delta(q, a) \setminus \{q_0\}$ for all $q \in Q'$ and $a \in \Sigma$. For each $q \in Q'$, we define an NFA $N_q = (Q', \Sigma, \delta', q, F')$, and we write $A_q = L(N_q)$. Applying the induction hypothesis to N_q , we conclude that A_q is bounded.

The key observation is that $L(M) = L_1 \cup L_2$, where

$$L_1 = \bigcup_{a \in \Sigma} \left(\bigcup_{q \in \delta(q_0, a)} L_{q_0} a A_q \right),$$

and

$$L_2 = \begin{cases} L_{q_0}, & \text{if } q_0 \in F; \\ \emptyset, & \text{if } q_0 \notin F. \end{cases}$$

By assumption, $L_{q_0} \subseteq u^*$ for some $u \in \Sigma^*$, and, as previously noted, by the induction hypothesis each of the languages A_q is bounded. It follows that $L(M)$ is a finite union of bounded languages, and hence is itself bounded. We conclude that $L(M)$ has polynomial growth, as required. \square

We now are ready to prove Theorem 2.

Proof. Let n denote the number of states of M . The idea is as follows. For every $q \in Q$, if L_q is commutative, then there exists $u \in \Sigma^*$ such that $L_q \subseteq u^*$. For any $w \in L_q$, we thus have $w \in u^*$. If z is the primitive root of w , then z is also the primitive root of u . If $L_q \subseteq z^*$, then L_q is commutative. On the other hand, if $L_q \not\subseteq z^*$, then L_q contains two words with different primitive roots, and is thus not commutative. This argument leads to the following algorithm.

For each $q \in Q$ we perform the following steps.

- Construct the NFA M_q accepting L_q . This takes $O(n + t)$ time.
- Find a word $w \in L(M_q)$, where $|w| < n$. If $L(M_q)$ is non-empty, such a w exists and can be found in $O(n + t)$ time.
- Find the primitive root of w , i.e., the shortest word z such that $w = z^k$ for some $k \geq 1$. This can be done in $O(n)$ time using the Knuth–Morris–Pratt algorithm. To find the primitive root of $w = w_1 \cdots w_\ell$, use Knuth–Morris–Pratt to find the first occurrence of w in $w_2 \cdots w_\ell w_1 \cdots w_{\ell-1}$. If the first occurrence begins at position i , then $z = w_1 \cdots w_{i-1}$ is the primitive root of w .

- Apply the cross product construction to obtain an NFA M' that accepts $L_q \setminus z^*$. The NFA M' has $O(n^2)$ states and $O(nt)$ transitions.
- Test whether $L(M')$ is empty or not. If $L(M')$ is non-empty, then by Lemma 3 the growth of $L(M)$ is exponential. If $L(M')$ is empty, then L_q is commutative. This step takes $O(n^2 + nt)$ time.

If for all $q \in Q$ we have verified that L_q is commutative, then by Lemma 4 $L(M)$ has polynomial growth.

The runtime of this algorithm is $O(n^3 + n^2t)$. □

3 Finding the exact order of polynomial growth

In this section we show that given a DFA accepting a language of polynomial growth, it is possible to efficiently determine the exact order of polynomial growth. We give two different algorithms: one combinatorial, the other algebraic.

Szillard et al. [16, Theorem 4] proved a weaker result: namely, that given a regular language L and an integer $d \geq 0$ it is decidable whether L has $O(m^d)$ growth. However, even if L is specified by a DFA, their algorithm takes exponential time.

3.1 A combinatorial algorithm

Theorem 5. *Given a DFA M with n states such that $L(M)$ is of polynomial growth, it is possible to determine the exact order of polynomial growth in $O(n^2)$ time.*

Proof. Let $M = (Q, \Sigma, \delta, q_0, F)$. Again we assume that every state of M is both accessible and co-accessible. Since $L(M)$ is of polynomial growth, by Lemma 3 M has the property that for every $q \in Q$ there exists $u \in \Sigma^*$ such that $L_q \subseteq u^*$.

Since M is deterministic, for any state q of M , if there exists a non-empty word w that takes M from state q back to q , the smallest such word w is unique. There is also a unique cycle of states of M associated with such a word w , and all such cycles in M are disjoint.

We now *contract* (in the standard graph-theoretical sense) each such cycle to a single vertex and mark this vertex as *special*. If any vertex on a contracted cycle was final, we also mark the new special vertex as final. Since all the cycles in M are disjoint, after contracting all of them, the transition graph of the automaton M now becomes a *directed acyclic graph* (DAG) D . A path in D from the start vertex to a final vertex that visits special vertices Q_1, Q_2, \dots, Q_k corresponds to a family of words in $L(M)$ of the form

$$x_1 y_1^* x_2 y_2^* \cdots x_k y_k^* x_{k+1}, \tag{1}$$

where the y_i 's are words labeling the cycles in M corresponding to the Q_i 's in D . Note that if a cycle in M is of size t , there could be up to t possible choices for the corresponding y_i .

There are only finitely many paths in D , and only finitely many choices for the x_i 's and y_i 's in a decomposition of the form given by (1). It follows that $L(M)$ is a finite union of

languages of the form $x_1 y_1^* x_2 y_2^* \cdots x_k y_k^* x_{k+1}$. We have thus recovered the characterization of Szilard et al. [16]. It is well-known that any language of this form has $O(m^{k-1})$ growth (see, for example, [16, Lemma 4]).

Consider a path through D from the start vertex to a final vertex that visits the maximum number d of special vertices. Then we may conclude that the order of growth of $L(M)$ is $\Theta(m^{d-1})$. This observation leads to our desired algorithm.

We first identify all the cycles in M and contract them to obtain a DAG D , as previously described. It remains to find a path through D from the start vertex to a final vertex that visits the largest number of special vertices. The LONGEST PATH problem for general graphs is NP-hard; however, in the case of a DAG, it can be solved in linear time by a simple dynamic programming algorithm. To obtain our result, we modify this dynamic programming algorithm by adjusting our distance metric so that the length of a path is not the number of edges on it, but rather the number of special vertices on the path. The most computationally intensive part of this algorithm is finding and contracting the cycles in M , which can be done in $O(n^2)$ time. \square

3.2 An algebraic approach

We now consider an algebraic approach to determining whether the order of growth is polynomial or exponential, and in the polynomial case, the order of polynomial growth. Let $M = (Q, \Sigma, \delta, q_0, F)$, where $|Q| = n$, and let $A = A(M) = (a_{ij})_{1 \leq i, j \leq n}$ be the *adjacency matrix* of M , that is, a_{ij} denotes the number of paths of length 1 from q_i to q_j . Then $(A^m)_{i,j}$ counts the number of paths of length m from q_i to q_j . Since a final state is reachable from every state q_j , the order of growth of $L(M)$ is the order of growth of A^m as $m \rightarrow \infty$. This order of growth can be estimated using nonnegative matrix theory.

Theorem 6 (Perron-Frobenius). *Let A be a nonnegative square matrix, and let r be the spectral radius of A , i.e., $r = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$. Then*

1. r is an eigenvalue of A ;
2. there exists a positive integer h such that any eigenvalue λ of A with $|\lambda| = r$ satisfies $\lambda^h = r^h$.

For more details, see [13, Chapters 1, 3].

Definition 1. The number $r = r(A)$ described in the above theorem is called the *Perron-Frobenius eigenvalue* of A . The *dominating Jordan block* of A is the largest block in the Jordan decomposition of A associated with $r(A)$.

Lemma 7. *Let A be a nonnegative $n \times n$ matrix over the integers. Then either $r(A) = 0$ or $r(A) \geq 1$.*

Proof. Let $r(A) = r$, $\lambda_1, \dots, \lambda_\ell$ be the distinct eigenvalues of A , and suppose that $r < 1$. Then $\lim_{m \rightarrow \infty} r^m = \lim_{m \rightarrow \infty} \lambda_i^m = 0$ for all $i = 1, \dots, \ell$, and so $\lim_{m \rightarrow \infty} A^m = 0$ (the zero matrix). But A^m is an integral matrix for all $m \in \mathbb{N}$, and the above limit can hold if and only if A is nilpotent, i.e., $r = \lambda_i = 0$ for all $i = 1, \dots, \ell$. \square

Lemma 8. *Let A be a nonnegative $n \times n$ matrix over the integers. Let $r(A) = r, \lambda_1, \dots, \lambda_\ell$ be the distinct eigenvalues of A , and let d be the size of the dominating Jordan block of A . Then $A^m \in \Theta(r^m m^{d-1})$.*

Proof. The theorem trivially holds for $r = 0$. Assume $r \geq 1$. Without loss of generality, we can assume that A does not have an eigenvalue λ such that $\lambda \neq r$ and $|\lambda| = r$; if such an eigenvalue exists, replace A by A^h (see Theorem 6). Let J be the Jordan canonical form of A , i.e., $A = SJS^{-1}$, where S is a nonsingular matrix, and J is a diagonal block matrix of Jordan blocks. We use the following notation: $J_{\lambda,e}$ is a Jordan block of order e corresponding to eigenvalue λ , and O_x is a square matrix, where all entries are zero, except for x at the top-right corner. Let $J_{r,d}$ be the dominating Jordan block of A . It can be verified by induction that

$$J_{r,d}^m = \begin{pmatrix} r^m & \binom{m}{1}r^{m-1} & \binom{m}{2}r^{m-2} & \cdots & \binom{m}{d-2}r^{m-d+2} & \binom{m}{d-1}r^{m-d+1} \\ 0 & r^m & \binom{m}{1}r^{m-1} & \cdots & \binom{m}{d-3}r^{m-d+3} & \binom{m}{d-2}r^{m-d+2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r^m & \binom{m}{1}r^{m-1} \\ 0 & 0 & 0 & \cdots & 0 & r^m \end{pmatrix}.$$

Thus the first row of $J_{r,d}^m$ has the form

$$r^m \begin{bmatrix} 1 & \frac{m}{r} & \frac{m(m-1)}{2!r^2} & \cdots & \frac{m(m-1) \cdots (m-(d-2))}{(d-1)!r^{d-1}} \end{bmatrix},$$

and so

$$\lim_{m \rightarrow \infty} \frac{J_{r,d}^m}{r^m m^{d-1}} = O_\alpha, \quad \text{where } \alpha = \frac{1}{(d-1)!r^{d-1}}.$$

All Jordan blocks other than the dominating block converge to zero blocks. and

$$\lim_{m \rightarrow \infty} \frac{A^m}{r^m m^{d-1}} = S \lim_{m \rightarrow \infty} \frac{J^m}{r^m m^{d-1}} S^{-1}.$$

The result follows. □

Note: The growth order of A^m supplies an algebraic proof of the fact that regular languages can grow either polynomially or exponentially, but no intermediate growth order is possible. This result can also be derived from a more general matrix theoretic result of Bell [1].

Lemma 8 implies that to determine the order of growth of $L(M)$, we need to compute the Perron-Frobenius eigenvalue r of $A(M)$: if $r = 0$, then $L(M)$ is finite; if $r = 1$, the order of growth is polynomial; if $r > 1$, the order of growth is exponential. In the polynomial case, if we want to determine the order of polynomial growth, we need to also compute the size of the dominating Jordan block, which is the algebraic multiplicity of r in the minimal polynomial of $A(M)$.

Both computations can be done in polynomial time, though the runtime is more than cubic. The characteristic polynomial, $c_A(x)$, can be computed in $\tilde{O}(n^4 \log \|A\|)$ bit operations

(here \tilde{O} stands for soft- O , and $\|A\|$ stands for the L_∞ norm of A). If $c_A(x) = x^n$ then $r = 0$; else, if $c_A(1) \neq 0$, then $r > 1$. In the case of $c_A(1) = 0$, we need to check whether $c_A(x)$ has a real root in the open interval $(1, \infty)$. This can be done using a real root isolation algorithm; it seems the best deterministic one uses $\tilde{O}(n^6 \log^2 \|A\|)$ bit operations [3]. The minimal polynomial, $m_A(x)$, can be computed through the rational canonical form of A in $\tilde{O}(n^5 \log \|A\|)$ bit operations (see references in [4]). All algorithms mentioned above are deterministic; both $c_A(x)$ and $m_A(x)$ can be computed in $\tilde{O}(n^{2.697263} \log \|A\|)$ bit operations using a randomized Monte Carlo algorithm [10].

An interesting problem is the following: given a nonnegative integer matrix A , is it possible to decide whether $r(A) > 1$ in time better than $\tilde{O}(n^6 \log^2 \|A\|)$? Using our combinatorial algorithm, we can do it in time $O(n^4 \|A\|)$, by interpreting A as the adjacency matrix of a DFA over an alphabet of size $\|A\|$, and applying the algorithm to each of the connected components of A separately. It would be interesting to find an algorithm polynomial in $\log \|A\|$.

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References

- [1] J. Bell, “A gap result for the norms of semigroups of matrices”, *Linear Algebra Appl.* **402** (2005), 101–110.
- [2] M. Bridson, R. Gilman, “Context-free languages of sub-exponential growth”, *J. Comput. System Sci.* **64** (2002), 308–310.
- [3] A. Eigenwillig, V. Sharma, C. K. Yap, “Almost tight recursion tree bounds for the Descartes method”, In *ISSAC '06: Proceedings of the 2006 International Symposium on Symbolic and Algebraic Computation*, pp. 71–78, 2006.
- [4] M. Giesbrecht, A. Storjohann, “Computing rational forms of integer matrices”, *J. Symbolic Comput.* **34** (2002), 157–172.
- [5] S. Ginsburg, *The Mathematical Theory of Context-free Languages*, McGraw–Hill, 1966.
- [6] S. Ginsburg, E. Spanier, “Bounded ALGOL-like languages”, *Trans. Amer. Math. Soc.* **113** (1964), 333–368.
- [7] S. Ginsburg, E. Spanier, “Bounded regular sets”, *Proc. Amer. Math. Soc.* **17** (1966), 1043–1049.

- [8] L. Ilie, G. Rozenberg, A. Salomaa, “A characterization of poly-slender context-free languages”, *Theoret. Informatics Appl.* **34** (2000), 77–86.
- [9] R. Incitti, “The growth function of context-free languages”, *Theoret. Comput. Sci.* **225** (2001), 601–605.
- [10] E. Kaltofen, G. Villard, “On the complexity of computing determinants”, *Comput. Complex.* **13** (2004), 91–130.
- [11] M. Latteux, G. Thierrin, “On bounded context-free languages”, *Elektron. Informationsverarb. Kybernet.* **20** (1984), 3–8.
- [12] R. C. Lyndon, M.-P. Schützenberger, “The equation $a^M = b^N c^P$ in a free group”, *Michigan Math. J.* **9** (1962), 289–298.
- [13] H. Minc, *Nonnegative Matrices*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley and Sons Inc., 1988.
- [14] O. Ibarra, B. Ravikumar, “On sparseness, ambiguity and other decision problems for acceptors and transducers”. In *Proc. STACS 1986*, pp. 171–179, LNCS 210, Springer, 1986.
- [15] D. Raz, “Length considerations in context-free languages”, *Theoret. Comput. Sci.* **183** (1997), 21–32.
- [16] A. Szilard, S. Yu, K. Zhang, J. Shallit, “Characterizing regular languages with polynomial densities”. In *Proc. MFCS 1992*, pp. 494–503, LNCS 629, Springer, 1992.
- [17] V. I. Trofimov, “Growth functions of some classes of languages”, *Cybernetics* (1981), no. 6, i, 9–12, 149.